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# REPRESENTATIONS OF ANISOTROPIC UNITARY GROUPS

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ABSTRACT. Let $SU(f)$ be the special unitary group of an anisotropic hermitian form $f$ over a field $k$. Assume $f$ represents only one norm class in $k$. The representations $\alpha\colon SU(f) \to SL(n,R)$ are characterized when $R$ is a commutative ring with 2 not a zero divisor and $n = \dim f \geq 3$ with $n \neq 4, 6$. In particular, a partial classification of the normal subgroups of $SU(f)$ is given when $k$ is the function field $\mathbf{C}(X)$.

**1. Introduction.** We study representations $\alpha\colon U(f) \to GL(n,R)$ where $U(f)$ is the unitary or orthogonal group of an anisotropic form $f$ over a field $k$ with $\dim f = n \geq 3$ and $R$ is a commutative ring with 2 not a zero divisor. Representations of the special unitary group $SU(f)$ will also be considered and, in particular, a partial classification of the normal subgroups of $SU(f)$ is obtained. However, the method only applies to a restricted class of fields $k$, including all real closed fields. The results are first established for $R$ a local ring using a generalization of the fundamental theorem of projective geometry to construct a generalized place $\varphi\colon k \to R \cup \infty$ with the inverse image $A = \varphi^{-1}(R)$ a valuation ring of $k$. The kernel of the homomorphism $\varphi\colon A \to R$ is now an ideal $\mathfrak{a}$ of $A$ which need not be maximal. The kernel of the homomorphism $\alpha$ is a twisted congruence subgroup $U(\mathfrak{a}, \chi)$ of $U(f)$ defined with respect to $\mathfrak{a}$ and a character $\chi\colon U(f) \to \mathfrak{u}(R)$, where $\mathfrak{u}(R)$ denotes the units of $R$. For general commutative rings $R$, the kernel of $\alpha$ is the intersection of the local kernels obtained after $\alpha$ is extended by localizing at the maximal ideals of $R$. These results generalize earlier work of Weisfeiler [**16**] and James [**9**, **10**] where $R$ is a field. Earlier, Borel and Tits [**3**] had studied abstract homomorphisms of isotropic algebraic groups and Tits [**15**] had considered representations of Lie groups. See [**8**] for a general survey of this area.

Let $k$ be a field of characteristic not two with involution $^*$, $V$ a $k$-space of finite dimension $n \geq 3$ and $f\colon V \times V \to k$ an anisotropic hermitian form. Thus $f(x,x) = 0$ implies $x = 0$. Let $U(f)$ be the unitary group of $f$ and $I(f)$ the subgroup of $U(f)$ generated by involutions. We allow the involution $^*$ on $k$ to be trivial, in which case $f$ becomes a quadratic form and $U(f) = I(f)$ is an orthogonal group. The symmetry $\Psi(x)\colon y \mapsto y - 2f(x,y)f(x,x)^{-1}x$ is an involution in $U(f)$ for each nonzero $x$ in $V$. Henceforth we only consider forms $f$ for which all symmetries in $U(f)$ are conjugate under $SI(f) = I(f) \cap SU(f)$. By Witt's Theorem this means that $f$ can only represent one norm class (or square class when $^*$ is trivial). Then necessarily $f$ can be diagonalized as a sum of norms (or squares) and, since $f$ is anisotropic, $k$ will have characteristic zero. If $f$ is a quadratic form then $k$ must be

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pythagorean and formally real (see Lam [**12**]); this includes all real closed fields, as well as fields of Laurent series in several variables over $\mathbf{R}$ or any other real closed field. When $f$ is hermitian there are more possibilities. Now we can choose $k = K(a)$, with $K$ formally real pythagorean and $a \notin K$ with $a^2 \in K$, or $K$ any formally real field with pythagoras number two and $a^2 = -1$. In particular, our results now apply to the function field $\mathbf{C}(X)$ and this case will be considered in greater detail. Some of our results will also hold when $f$ is an hermitian form over a quaternion division algebra of special type (see [**10**, p. 348]).

It is shown in [**10**, Proposition 3.1] that $U(f)$ is generated by $I(f)$ and any one-dimensional subgroup $U(1)$ of $U(f)$. Since $SI(f)$ is generated by products of two symmetries $\Psi(x)\Psi(y)$ where $\Psi(x)$ and $\Psi(y)$ are conjugate, it follows that $SI(f)$ is contained in the commutator subgroup $[U(f), U(f)]$. If $U(1)$ is the one-dimensional subgroup on the subspace $kx$ of $V$ then $\Psi(x) \in U(1)$. Hence the map $\det\colon U(f) \to k^*$ has kernel $SI(f)$. Thus $SI(f) = SU(f)$ is the commutator subgroup of $U(f)$ and

$$1 \to SU(f) \to U(f) \xrightarrow{\det} N_1(k) \to 1$$

is an exact sequence, where $N_1(k) = \{a \in k | a^* a = 1\}$ is the norm one subgroup of $k^*$. It is also easily seen that $SU(f)$ is perfect.

Our main result is the following.

THEOREM 1. *Let $f$ be an anisotropic hermitian form which represents only one norm class and let $G$ be a subgroup of $U(f)$ containing $SU(f)$. Let $\alpha\colon G \to GL(M)$ be a representation of $G$ where $M$ is a free module over a commutative local ring $R$ with 2 a unit and $\alpha SU(f) \neq 1$. Assume* $\dim f = \operatorname{rank} M = n \geq 3$ *and $n \neq 4, 6$. Then there exist*

(i) *a generalized place $\varphi\colon k \to R \cup \infty$ with $A = \varphi^{-1}(R) = A^*$ a valuation ring and $\varphi\colon A \to R$ a homomorphism with kernel $\mathfrak{a}$,*

(ii) *a free maximal $A$-lattice $N$ in $V$ of rank $n$ with $U(N) = U(f)$,*

(iii) *a $\varphi$-semilinear map $\beta\colon N \to M$ with kernel $\mathfrak{a}N$,*

(iv) *a character $\chi\colon G \to \mathfrak{u}(R)$,*

(v) *a twisted congruence subgroup $G(\mathfrak{a}, \chi)$ of $G$,*

*such that*

$$1 \to G(\mathfrak{a}, \chi) \to G \xrightarrow{\alpha} GL(M)$$

*is an exact sequence. Moreover, for each $\sigma \in G$ and $x \in N$, $(\alpha\sigma)\beta(x) = \chi(\sigma)\beta\sigma(x)$. In particular, when $G = SU(f)$, the diagram*

$$\begin{array}{ccc} N & \xrightarrow{\sigma} & N \\ \beta\big\downarrow & & \big\downarrow\beta \\ M & \xrightarrow{\alpha\sigma} & M \end{array}$$

*is commutative.*

The theorem is false when $n = 4$ and $G = SU(f)$, see Artin [**1**, Theorem 5.23] since there are now nontrivial homomorphisms $\alpha\colon SO(f) \to SL(3, k)$ with kernel not a congruence subgroup. Probably the theorem is true when $n = 6$, but there are some technical difficulties at one point in our method of proof. However, if we modify the assumptions on $G$ to $I(f) \subseteq G$ and $\operatorname{card} \alpha I(f) > 2$, then the theorem remains true for both $n = 4$ and 6.

Theorem 1 can be used, with the help of a localization argument, to give the following partial classification of the normal subgroups of $SU(f)$ in terms of congruence subgroups.

THEOREM 2. *Let $f$ be an anisotropic hermitian form which represents only one norm class. Assume* $\dim f = n \geq 3$ *and* $n \neq 4, 6$. *Let $H$ be a nontrivial normal subgroup of $SU(f)$ such that $SU(f)/H$ is contained in $GL(n, R)$ for some commutative ring $R$ with 2 not a zero divisor. Then $H$ is the intersection of a family of congruence subgroups of the type $SU(\mathfrak{a}, 1)$ in Theorem* 1.

Special cases of Theorem 2 are known for orthogonal groups without the restriction $SO(f)/H \subseteq GL(n, R)$. When $k = \mathbf{R}$ the projective group $PSO(f)$ is simple, $n \neq 4$ (see [**1**]). Pollak [**14**] considered the field $k = \mathbf{R}((X))$ of formal Laurent series and Chang [**5**] generalized this to several variables. Archimedean ordered pythagorean fields are considered by Bröcker [**4**]. Probably their results can also be modified to unitary groups.

For specific fields $k$ the results in Theorems 1 and 2 can be strengthened. In particular, if $\mathbf{R}$ is contained in the fixed field of $k$, then the valuation on $\mathbf{R}$ induced by the restricted place is trivial (Proposition 8).

We will consider unitary groups over the rational function field $k = \mathbf{C}(X)$, with involution induced by $i^* = -i$, in detail. For each complex number $c$ the identity map $\varphi_c \colon \mathbf{C} \to \mathbf{C}$ extends to a unique place $\varphi_c \colon k \to \mathbf{C} \cup \infty$ by setting $\varphi_c(X) = c$. The associated valuation ring $A_c$ is the set of rational functions $r(X) = g(X)/h(X)$ over $\mathbf{C}$ with $g(X), h(X)$ relatively prime polynomials in $\mathbf{C}[X]$ and $h(c) \neq 0$. Fix an integer $m \geq 1$ and let $R(c, m)$ denote the local ring $\mathbf{C}[X]/(X-c)^m$. Then a generalized place $\varphi_{c,m} \colon k \to R(c, m) \cup \infty$ can be obtained as follows. If $r(X) \notin A_c$ set $\varphi_{c,m}(r(X)) = \infty$. Any $r(X) \in A_c$ can be expanded as a unique formal power series $r(X) = \sum_{j=0}^{\infty} a_j (X-c)^j$ with $a_j \in \mathbf{C}$. Set $\varphi_{c,m}(r(X)) = \sum_{j=0}^{m-1} a_j (X-c)^j$, viewed as an element in $R(c, m)$. The kernel of $\varphi_{c,m} \colon A_c \to R(c, m)$ is the ideal $(X-c)^m A_c$. Another place $\varphi_\infty$ on the field $\mathbf{C}(X)$ can be obtained by setting $\varphi_\infty(X) = \infty$; the associated valuation ring $A_\infty$ consists of all rational functions $r(X)$ with $\deg(r(X)) \leq 0$. For each integer $m \geq 1$ there corresponds a generalized place $\varphi_{\infty,m}$. The kernel of the restriction of $\varphi_{\infty,m}$ to $A_\infty$ consists of all rational functions $r(X)$ with $\deg(r(X)) \leq -m$.

Now let $A$ be the intersection of a finite number of the valuation rings of $\mathbf{C}(X)$ just considered with $\varphi(X)$ real or infinite. Then $A$ is a semilocal ring which can be described as follows. Choose a finite number of real numbers $c_1, \ldots, c_l$ (possibly none) and also possibly choose $\infty$. Then $A$ consists of all rational functions $r(X) = g(X)/h(X)$ with $h(c_i) \neq 0$, $1 \leq i \leq l$, and also $\deg(r(X)) \leq 0$ if $\infty$ is chosen. Let $V$ be a vector space over $\mathbf{C}(X)$ with basis $u_1, \ldots, u_n$, $n \geq 3$, and let $f$ be the anisotropic hermitian form defined by $f(u_i, u_j) = \delta_{ij}$, $1 \leq i, j \leq n$. Then $N = Au_1 \perp \cdots \perp Au_n$ is a maximal lattice in $V$ consisting of all $x \in V$ with $f(x, x) \in A$ (if any $c_i$ is nonreal, $N$ is not maximal). Hence $U(N) = U(f)$ where $U(N)$ consists of all $\sigma \in U(f)$ with $\sigma(N) = N$. Let $\mathfrak{a}$ be an ideal of the principal ideal ring $A$ and define $SU(\mathfrak{a})$ as the congruence subgroup of $SU(f)$ consisting of all $\sigma \in SU(f)$ for which $(\sigma - 1_V)(N) \subseteq \mathfrak{a}N$. Then we have an exact sequence

$$1 \to SU(\mathfrak{a}) \to SU(f) \to SL(n, R)$$

where $R$ is the commutative ring $A/\mathfrak{a}$. The ideal $\mathfrak{a}$ is generated by a rational function $r(X)$ of the form

$$r(X) = (X - c_1)^{m_1} \cdots (X - c_l)^{m_l}/(X - c)^{m+m_1+\cdots+m_l}$$

with $c_1, \ldots, c_l, c$ all real and distinct; this gives the connection with the generalized places.

THEOREM 3. *Let $H$ be a nontrivial normal subgroup of the special unitary group $SU(f)$ over the field $\mathbf{C}(X)$. Assume $n = \dim f \geq 3$, $n \neq 4, 6$, and $SU(f)/H \subseteq GL(n, R)$ for some commutative ring $R$ with 2 not a zero divisor. Then $H$ is a congruence subgroup $SU(\mathfrak{a})$.*

The above three theorems are a further contribution to the solutions of very general questions asked by Weisfeiler [**11**, XVII and **17**, §7], although we have gone beyond his original framework in allowing the image group to be defined over a ring.

**2. Collineations and projective geometry.** Let $R$ be a local ring with maximal ideal $\mathfrak{m} = \mathfrak{m}(R)$. In this section it is not necessary to assume that $R$ is commutative or that 2 is a unit. Let $M$ be a free $R$-module of finite rank $n \geq 3$. A *point* of $M$ is a rank one direct summand of $M$ and a *line* is a rank two direct summand of $M$. The *projective space* $\mathbf{P}M$ of $M$ is the set of all points of $M$. A *projective frame* of $\mathbf{P}M$ is a set of $n$ points $P_1, \ldots, P_n$ such that $M = P_1 + P_2 + \cdots + P_n$. A special case of the above situation is a vector space $V$ over a (skew) field $k$ with associated projective space $\mathbf{P}V$.

DEFINITION. A *collineation* (or *projective homomorphism*) is a map $\pi\colon \mathbf{P}V \to \mathbf{P}M$ sending points of $\mathbf{P}V$ to points of $\mathbf{P}M$ such that

(i) there exists a projective frame of $\mathbf{P}V$ which is carried by $\pi$ to a projective frame of $\mathbf{P}M$ (and hence $\dim V = \operatorname{rank} M$),

(ii) if the images of the points $P_1$ and $P_2$ of $\mathbf{P}V$ generate a line $\pi P_1 + \pi P_2$ in $\mathbf{P}M$, then any point $P$ on the line $P_1 + P_2$ has image $\pi P$ on the line $\pi P_1 + \pi P_2$.

If, moreover, in (ii) above, there always exists a point $P_3$ on the line $P_1 + P_2$ such that $\pi P_1 + \pi P_2 = \pi P_1 + \pi P_3 = \pi P_2 + \pi P_3$, then the collineation $\pi$ is called *thick*.

If $P_1$ and $P_2$ are distinct points in $\mathbf{P}V$ then $P_1 + P_2$ is always a line. However, this is false in $\mathbf{P}M$ since the span of the points need not be a direct summand of $M$. This is the reason for the more careful phrasing in the definition of a collineation.

FUNDAMENTAL THEOREM. *Let $\pi\colon \mathbf{P}V \to \mathbf{P}M$ be a thick collineation. Then there exist*

(i) *a generalized place $\varphi\colon k \to R \cup \infty$ with $A = \varphi^{-1}(R)$ a valuation ring of $k$ and the restriction $\varphi\colon A \to R$ a ring homomorphism,*

(ii) *a free $A$-module $N$ in $V$ with* $\operatorname{rank} N = \dim V$,

(iii) *a $\varphi$-semilinear map $\beta\colon N \to M$,*

*such that $\beta$ induces $\pi$, namely, $\pi P = R\beta(P \cap N)$ for $P \in \mathbf{P}V$.*

PROOF. Choose a basis $u_1, \ldots, u_n$ in $V$ such that $\pi(ku_i) = Rv_i$, $1 \leq i \leq n$, is a projective frame for $\mathbf{P}M$. Thus $M = Rv_1 + \cdots + Rv_n$. Fix $j$ with $2 \leq j \leq n$.

Since $Rv_1$ and $Rv_j$ span a line, it follows from the definition of a thick collineation that there exists a point $P_3 = k(u_1 + au_j)$ in $\mathbf{P}V$, $a \neq 0$, with image $R(bv_1 + cv_j)$ satisfying

$$Rv_1 + Rv_j = Rv_1 + R(bv_1 + cv_j) = Rv_j + R(bv_1 + cv_j).$$

Hence $b$ is a unit in $R$ and we may assume $b = 1$. Also $c$ is a unit in $R$. Replacing $u_j$ by $a^{-1}u_j$ and $v_j$ by $c^{-1}v_j$ we can normalize our choice of basis $v_1, \ldots, v_n$ for $M$ such that

$$\pi(k(u_1 + u_j)) = R(v_1 + v_j), \qquad 2 \leq j \leq n.$$

For each $a \in k$ the point $k(u_1 + au_j)$, $j \neq 1$, lies on the line $ku_1 + ku_j$ and hence its image is either $R(v_1 + bv_j)$ for some $b \in R$, or $R(pv_1 + v_j)$ for some $p \in \mathfrak{m}$. In the first case set $\varphi_j(a) = b$ and in the second case set $\varphi_j(a) = \infty$. Then for each $j \neq 1$ we have defined a map $\varphi_j \colon k \to R \cup \infty$ with $\varphi_j(0) = 0$ and $\varphi_j(1) = 1$. Let $A_j = \varphi_j^{-1}(R)$. We now prove that $\varphi_j = \varphi$ and $A_j = A$ are independent of $j$, that $A$ is a valuation ring of $k$ and $\varphi \colon A \to R$ is a ring homomorphism.

Fix $1 < i \neq j \leq n$ and let $a \in A_i$ and $b \in A_j$. Then by standard arguments it follows that the images of the points $k(u_1 + au_i + bu_j)$ and $k(au_i + u_j)$ are respectively, $R(v_1 + \varphi_i(a)v_i + \varphi_j(b)v_j)$ and $R(\varphi_i(a)v_i + v_j)$. Hence, for nonzero $a, a'$ in $A_i$, the image of the point $k(u_1 + (a + a')u_i + u_j)$ lies on both of the lines $R(v_1 + \varphi_i(a)v_i) + R(\varphi_i(a')v_i + v_j)$ and $R(v_1 + \varphi_i(a)v_i + v_j) + Rv_i$. If $\varphi_i(a + a') = \infty$ then this image also lies on the line $R(pv_1 + v_i) + Rv_j$ where $p \in \mathfrak{m}$, which leads to a contradiction. Thus $\varphi_i(a + a') \neq \infty$ and $A_i$ is closed under addition. It now easily follows that $\varphi_i(a + a') = \varphi_i(a) + \varphi_i(a')$.

Again take nonzero $a \in A_j$, $b \in A_i$. Since the images of $ku_1$ and $k(bu_i + u_j)$ span a line in $\mathbf{P}M$, as do the images of $k(u_1 + au_j)$ and $ku_i$, it follows that the image of $k(u_1 + abu_i + au_j)$ is $R(v_1 + \varphi_j(a)\varphi_i(b)v_i + \varphi_j(a)v_j)$. Hence the image of $k(u_1 + abu_i)$ cannot be $R(pv_1 + v_i)$ and so necessarily $ab \in A_i$. Letting $b = 1$ it follows that $A_j \subseteq A_i$ and hence $A_i = A_j = A$ is closed under multiplication. Moreover, $\varphi_i(ab) = \varphi_j(a)\varphi_i(b)$ so that $\varphi_i = \varphi_j = \varphi$ and $\varphi(ab) = \varphi(a)\varphi(b)$. Note we still have not shown that $A$ is a ring, namely $-1 \in A$.

Next consider $a \notin A$. Then $\pi(k(u_1 + au_2)) = R(pv_1 + v_2)$ where $p \in \mathfrak{m}$. Repeat all the previous arguments with $u_1$ and $u_2$ interchanged; there is no need to change the normalization of the basis, however, since it is already established that

$$\pi(k(u_2 + u_i)) = R(v_2 + v_i) \quad \text{for } i \neq 2.$$

We obtain a new set $B \subseteq k$, closed under addition and multiplication, and a map $\psi \colon B \to R$. For $c \in A$ the image of $k(u_1 + u_2 + cu_3)$ is $R(v_1 + v_2 + \varphi(c)v_3)$. If $c \notin B$, then $k(u_2 + cu_3)$ has image $R(pv_2 + v_3)$ where $p \in \mathfrak{m}$ and consequently the image of $k(u_1 + u_2 + cu_3)$ must be contained on the line $Rv_1 + R(pv_2 + v_3)$, which is a contradiction. Hence $A = B$ and $\varphi = \psi$. Again consider $a \notin A$. If $a^{-1} \notin A$, then $k(u_1 + au_2) = k(a^{-1}u_1 + u_2)$ forces $Rv_2 \equiv Rv_1 \bmod \mathfrak{m}$. Thus necessarily $a^{-1} \in A$; in particular $-1 = (-1)^{-1} \in A$. Hence $A$ is a valuation ring and $\varphi \colon A \to R$ is a ring homomorphism.

Finally, define $N = Au_1 + \cdots + Au_n$, a free $A$-module of rank $n$. Define $\beta \colon N \to M$ by $\beta(\sum a_i u_i) = \sum \varphi(a_i)v_i$. Then $\beta$ is a $\varphi$-semilinear map. Let $x = \sum c_i u_i \in V$ with $c_i \in k$ not all zero. Since $A$ is a valuation ring, there is a $j$ with $c_j^{-1}c_i \in A$ for all $i$. By standard arguments

$$\pi(kx) = R\left(\sum \varphi(c_j^{-1}c_i)v_i\right) = R\beta(kx \cap N)$$

completing the proof.

REMARK. Clearly $N$ is not unique since it depends on the choice of the projective frames at the beginning of the proof. However, if $k$ and $R$ are commutative, a standard argument shows that the generalized place $\varphi$ and its valuation ring $A$ are unique; in general they are only determined up to conjugacy. Also, $N$ and $\beta$ are determined up to multiplication by a nonzero element of $k$ and a unit of $R$, respectively. Note also that if $\varphi(a) = \infty$ then $\varphi(a^{-1}) \in \mathfrak{m}(R)$.

**3. Involutions.** Many of the results in this section are extensions of the work of Dieudonné [**6**, **7**]. Let $R$ be a local ring with 2 a unit, but not necessarily commutative, and $M$ a free $R$-module of finite rank $m \geq 3$. Let $\sigma$ be an involution in the general linear group $GL(M) = GL(m, R)$ and $P(\sigma) = \{x \in M | \sigma(x) = x\}$ and $N(\sigma) = \{x \in M | \sigma(x) = -x\}$ the positive and negative spaces of $\sigma$. Then $P(\sigma) \cap N(\sigma) = 0$ and $M = P(\sigma) + N(\sigma)$ is a direct sum so that both $P(\sigma)$ and $N(\sigma)$ are free modules. The involution $\sigma$ is called *extremal*, or a 1-involution, if either $P(\sigma)$ or $N(\sigma)$ has rank one. In general, $\sigma$ is an $l$-involution if either $P(\sigma)$ or $N(\sigma)$ has rank $l (\leq \frac{1}{2}m)$. Any set of mutually commuting involutions in $GL(M)$ contains at most $2^m$ elements and such a set can always be extended to a set of $2^m$ mutually commuting involutions in $GL(M)$ (see McDonald [**13**]). When $R$ is commutative, a set of mutually commuting involutions in the special linear group $SL(M)$ can always be extended to a maximal set of $2^{m-1}$ involutions. All of the above holds for involutions in the unitary group $U(f)$ of an anisotropic hermitian form $f$, although now we also have $V = P(\sigma) \perp N(\sigma)$ is an orthogonal sum.

PROPOSITION 1. *Let $f$ be an anisotropic hermitian form which represents only one norm class. Assume $n = \dim V$ is odd. Let $\alpha\colon SU(f) \to SL(m, R)$ be a nontrivial representation where $n \geq m \geq 3$ and $R$ is a commutative local ring with 2 a unit. Then $n = m$ and $\alpha$ preserves $l$-involutions for $1 \leq l < \frac{1}{2}n$.*

PROOF. Observe first, since $n$ is odd, that $-1_V \notin SU(f)$ and all extremal involutions in $SU(f)$ are of the form $-\Psi(x)$. Then $\alpha(-\Psi(x)) \neq 1_M$, for otherwise $\alpha$ is trivial. In fact, since any involution $\sigma$ in $SU(f)$ is a product of mutually commuting extremal involutions, it is easily seen that $\alpha(\sigma) \neq 1_M$ provided $\sigma \neq 1_M$. Any involution in $SU(f)$ can be embedded into a maximal set $S$ of $2^{n-1}$ mutually commuting involutions. Since $\operatorname{card} \alpha(S) = \operatorname{card} S = 2^{n-1}$, the image $\alpha(S)$ forms a set of $2^{n-1}$ mutually commuting involutions in $SL(m, R)$. Since $n \geq m$, this set is maximal and $n = m$. If $m = 3$ then $\alpha$ necessarily preserves extremal involutions. Hence we may assume $n \geq 5$. In $S$ there are exactly $n$ extremal involutions all conjugate under $SU(f)$. Any $l$-involution in $S$, where $1 < l < \frac{1}{2}n$, has at least $\frac{1}{2}n(n-1) > n$ conjugates in $S$. Since $\alpha\colon S \to SL(n, R)$ is injective and preserves conjugates, it follows that $\alpha$ preserves extremal involutions. Similarly, $\alpha$ preserves $l$-involutions.

REMARK. A slight modification of the above argument shows that a homomorphism $\alpha\colon U(f) \to GL(m, R)$ preserves extremal involutions for any $n \geq m \geq 3$ provided we assume $\operatorname{card} \alpha I(f) > 2$; also, it is now not necessary to assume $k$ and $R$ are commutative (see James [**9**, Proposition 2.7]).

We next consider the case $n$ even. Then $-1_V \in SU(f)$, but $SU(f)$ contains no extremal involutions.

PROPOSITION 2. *Let $f$ be an anisotropic hermitian form which represents only one norm class. Assume $n = \dim V \geq 6$ is even. Let $\alpha\colon SU(f) \to SL(m, R)$ be a nontrivial representation where $n \geq m \geq 3$ and $R$ is a commutative local ring with 2 a unit. Then $n = m$ and $\alpha$ preserves 2-involutions.*

PROOF. Any 2-involution in $SU(f)$ is of the form $\pm\Psi(x)\Psi(y)$ where $f(x, y) = 0$ and can be embedded into a maximal set $S$ of $2^{n-1}$ mutually commuting involutions of $SU(f)$. Since $\alpha SU(f) \neq 1_M$ no noncentral involution can be killed by $\alpha$. Hence the kernel of $\alpha\colon S \to SL(m, R)$ is contained in $\{\pm 1_V\}$. If $\alpha(-1_V) = -1_M$ then $\alpha(S)$ is a set of $2^{n-1}$ mutually commuting involutions in $SL(m, R)$ and hence $n = m$. If $\alpha(-1_V) = 1_M$ and $m$ is even then $\alpha(S)$ and $-1_M$ generate a set of $2^{n-1}$ mutually commuting involutions in $SL(m, R)$ and again $n = m$. If, however, $m$ is odd we can only conclude $m \geq n - 1$ since now $-1_M \notin SL(m, R)$. Assume, if possible, $n = m + 1 \geq 6$. Now any noncentral element of $S$ has at least $\frac{1}{2}n(n-1)$ conjugates in $S$ and hence its image in $\alpha(S)$ has at least $\frac{1}{4}n(n-1) > n$ conjugates in $\alpha(S)$. Hence no involution in $\alpha(S)$ is extremal. Thus $\alpha(S)$ is not a maximal set of mutually commuting involutions in $SL(m, R)$ and, consequently, $m \geq n$. In all situations we now have $m = n$.

If $n = 6$, then all noncentral involutions in $SL(6, R)$ are 2-involutions and hence $\alpha$ preserves 2-involutions. Assume, therefore, $n = m \geq 8$. Then any 2-involution in $S$ has exactly $\frac{1}{2}n(n-1)$ conjugates in $S$ while any other noncentral involution in $S$ has at least $\binom{n}{4} > n(n-1)$ conjugates in $S$. Since the kernel of $\alpha\colon S \to SL(n, R)$ has at most 2 elements, it again follows by counting conjugates that $\alpha$ preserves 2-involutions.

REMARK. Proposition 2 is false if $n = 4$ since there exist nontrivial representations of the type $\alpha\colon SO_4(f) \to SL(3, k)$ (see Artin [**1**, Theorem 5.23]). A slight modification of the above proof also shows that $\alpha$ preserves 4-involutions when $n \geq 8$.

The fact that $\alpha\colon SU(f) \to SL(n, R)$ preserves extremal involutions when $n$ is odd sets up a correspondence between points in $\mathbf{P}V$ and points in $\mathbf{P}M$ and hence defines a map $\pi\colon \mathbf{P}V \to \mathbf{P}M$. However, when $n$ is even the correspondence obtained from $\alpha$ is only between lines and we must get down to points by intersecting lines.

Assume $n \geq 8$ is even and let $\sigma, \tau$ be two commuting 2-involutions in $SL(n, R)$. Then $(\sigma, \tau)$ is called a *minimal pair* if $\sigma\tau$ is also a 2-involution. The same definition applies for involutions in $U(f)$. Denote by $L(\sigma)$ the rank two subspace of $M$ (either the positive or negative space) associated with the 2-involution $\sigma$. Then $L(\sigma)$ can be viewed as a line in $\mathbf{P}M$. If $(\sigma, \tau)$ is a minimal pair then $P(\sigma, \tau) = L(\sigma) \cap L(\tau)$ is a rank one direct summand of $M$ and hence a point in $\mathbf{P}M$. Note, if $\sigma$ and $\tau$ are commuting 2-involutions, then $\sigma\tau$ is either $\pm 1_M$, a 2-involution or a 4-involution, according as the rank of $L(\sigma) \cap L(\tau)$ is 2, 1 or 0. When $n = 6$, the 2-involutions and 4-involutions cannot be distinguished; hence the assumption $n \geq 8$. A homomorphism $\alpha\colon SU(f) \to SL(n, R)$ with nontrivial image will now map a minimal pair $(\sigma, \tau)$ in $SU(f)$ to a minimal pair $(\alpha\sigma, \alpha\tau)$ in $SL(n, R)$. We can then set up a map $\pi\colon \mathbf{P}V \to \mathbf{P}M$ by putting $\pi P(\sigma, \tau) = P(\alpha\sigma, \alpha\tau)$. That $\pi$ is well defined is an immediate consequence of the following result.

PROPOSITION 3. *Take the same assumptions as Proposition 2 except $n \geq 8$ and even. Let $\sigma_1, \sigma_2$ and $\sigma_3$ be three 2-involutions in $SU(f)$ whose associated lines in* $\mathbf{P}V$ *have a unique common point, that is, $L(\sigma_1) \cap L(\sigma_2) \cap L(\sigma_3) = kx$. Then the lines $L(\alpha\sigma_1)$, $L(\alpha\sigma_2)$ and $L(\alpha\sigma_3)$ have a common point, which is unique if $(\sigma_1, \sigma_2)$ is a minimal pair.*

PROOF. The subspace $U = L(\sigma_1) + L(\sigma_2) + L(\sigma_3)$ of $V$ has dimension at most four. Hence there exist nonzero $y_1, y_2$ in $V$, orthogonal to $U$, with $f(y_1, y_2) = 0$. Then $\tau_1 = \Psi(x)\Psi(y_1)$, $\tau_2 = \Psi(x)\Psi(y_2)$ and $\tau_3 = \tau_1\tau_2$ are mutually commuting 2-involutions in $SU(f)$. Any pair of $\tau_1, \tau_2, \tau_3$ form a minimal pair, in fact, $P(\tau_1, \tau_2) = kx$, $P(\tau_1, \tau_3) = ky_1$ and $P(\tau_2, \tau_3) = ky_2$. This constructs three points $P(\alpha\tau_1, \alpha\tau_2)$, $P(\alpha\tau_2, \alpha\tau_3)$ and $P(\alpha\tau_1, \alpha\tau_3)$ in $\mathbf{P}M$; these three points must be distinct for otherwise $\alpha(\tau_1\tau_2\tau_3) = \alpha(1_V)$ is a 4-involution, which is absurd. By construction $(\sigma_i, \tau_j)$, $1 \leq i \leq 3$, $1 \leq j \leq 2$, are minimal pairs; hence each $P(\alpha\sigma_i, \alpha\tau_j)$ is a point in $\mathbf{P}M$. Therefore, the line $L(\alpha\sigma_i)$ either contains the point $P(\alpha\tau_1, \alpha\tau_2) = L(\alpha\tau_1) \cap L(\alpha\tau_2)$, or lies in the plane (rank three direct summand) $L(\alpha\tau_1) + L(\alpha\tau_2)$ and hence $L(\alpha\sigma_i) \cap L(\alpha\tau_3) \neq 0$. But $\sigma_i\tau_3$ is a 4-involution. Hence $\alpha(\sigma_i\tau_3)$ is also a 4-involution and $L(\alpha\sigma_i) \cap L(\alpha\tau_3) = 0$. Thus necessarily the line $L(\alpha\sigma_i)$ goes through the point $P(\alpha\tau_1, \alpha\tau_2)$. This point need not be the unique point of intersection since the involutions $\alpha\sigma_i$ can all coincide. However, the additional assumption that $(\sigma_1, \sigma_2)$ is a minimal pair ensures that the point is unique.

**4. Thick collineations.** Let $f$ be an anisotropic hermitian form which represents only one norm class and $\alpha\colon SU(f) \to SL(n, R)$ a nontrivial homomorphism where $R$ is a local ring with 2 a unit and $\dim f = n \geq 3$. We now use the results from the previous selection to construct a thick collineation $\pi\colon \mathbf{P}V \to \mathbf{P}M$ from which, with the help of the Fundamental Theorem, the main results will be obtained.

Consider first the case $n$ odd. Then for each nonzero $x$ in $V$ we have $\alpha(-\Psi(x)) = E(X, y)$ where $E(X, y)$ is an extremal involution in $SL(n, R)$ with negative space $X$ and positive space $Ry$. Define the map $\pi\colon \mathbf{P}V \to \mathbf{P}M$ by $\pi(kx) = Ry$.

PROPOSITION 4. *Let $\alpha\colon SU(f) \to SL(n, R)$ be a nontrivial homomorphism with $n \geq 3$ odd. Then the map $\pi\colon \mathbf{P}V \to \mathbf{P}M$ induced by $\alpha$ is a collineation.*

PROOF. Let $u_1, \ldots, u_n$ be any orthogonal basis of $V$ and put $\alpha(-\Psi(u_i)) = E(U_i, v_i)$, $1 \leq i \leq n$. Since these involutions are mutually commutative it follows that $M = Rv_1 + \cdots + Rv_n$ and each $U_i = \sum_{j \neq i} Rv_j$ (see McDonald [**13**]). Hence the map $\pi$ sends the projective frame $ku_1, \ldots, ku_n$ of $\mathbf{P}V$ to the projective frame $Rv_1, \ldots, Rv_n$ of $\mathbf{P}M$ and consequently satisfies the first condition for a collineation.

Next let $P_1$ and $P_2$ be two points in $\mathbf{P}V$ whose images $\pi P_1$ and $\pi P_2$ span a line in $\mathbf{P}M$. We must prove that any point $P$ on the line $P_1 + P_2$ has image $\pi P$ on the line $\pi P_1 + \pi P_2$. Assume first that $f(P_1, P_2) = 0$. Let $P_1 = ku_1$, $P_2 = ku_2$ and expand $u_1, u_2$ to an orthogonal basis $u_1, u_2, \ldots, u_n$ of $V$. Put $\alpha(-\Psi(u_i)) = E(U_i, v_i) = E_i$, $1 \leq i \leq n$, as above. Let $P = kx$ and $\alpha(-\Psi(x)) = E(X, y)$ so that $\pi P = Ry$. We must prove $y \in Rv_1 + Rv_2$. For each $i \geq 3$ the commutator $[\Psi(x), \Psi(u_i)] = 1_V$ and hence $E_i E(X, y) = E(X, y) E_i$. It follows that $E(X, y)(v_i)$ lies in the positive space $Rv_i$ of $E_i$. If $E(X, y)(v_i) = v_i$ then $Ry = Rv_i$ and the positive space of the commuting involutions $E_i$ and $E(X, y)$ coincide, which forces the contradiction

$\alpha(\Psi(x)\Psi(v_i)) = 1_M$. Hence, necessarily $v_i$ lies in $X$, the negative space of $E(X,y)$, for $3 \leq i \leq n$. Let $y = \sum a_i v_i$ where $a_i \in R$. Then $E(X,y)E_i(y) = E_i E(X,y)(y) = E_i(y)$ and hence $E_i(y) \in Ry$, $3 \leq i \leq n$. Thus $cy = E_i(y) = 2a_i v_i - y$ for either $c = 1$ or $c = -1$. The first possibility again forces the contradiction $Ry = Rv_i$. Hence $c = -1$ and $a_i = 0$, $3 \leq i \leq n$. Thus $y \in Rv_1 + Rv_2$ and $\pi P$ lies on $\pi P_1 + \pi P_2$ as required.

Finally assume $f(P_1, P_2) \neq 0$. Then $P_1 + P_2 = ku_1' \perp ku_2'$ and the points $P_1' = ku_1'$ and $P_2' = ku_2'$ can be expanded to an orthogonal projective frame for $\mathbf{P}V$. As above $\pi P_1' + \pi P_2'$ is a line in $\mathbf{P}M$ and $\pi P_1$ and $\pi P_2$ must lie on this line. Hence $\pi P_1 + \pi P_2 = \pi P_1' + \pi P_2'$ since both are rank two direct summands of $M$. Consequently, the collinearity property also holds when $f(P_1, P_2) \neq 0$.

We next consider the case $n \geq 8$ even. Then, as explained in the previous section, a nontrivial homomorphism $\alpha\colon SU(f) \to SL(n,R)$ induces a map $\pi\colon \mathbf{P}V \to \mathbf{P}M$ via its action on minimal pairs of 2-involutions in $SU(f)$.

PROPOSITION 5. *Let $\alpha\colon SU(f) \to SL(n,R)$ be a nontrivial homomorphism with $n \geq 8$ even. Then the map $\pi\colon \mathbf{P}V \to \mathbf{P}M$ induced by $\alpha$ is a collineation.*

PROOF. Let $u_1, \ldots, u_n$ be an orthogonal basis of $V$ and put $\sigma_i = \Psi(u_i)\Psi(u_{i+1})$, $1 \leq i \leq n$, where we view the subscipts $i$ modulo $n$. Then each $(\sigma_{i-1}, \sigma_i)$ is a minimal pair in $SU(f)$ with associated point $P(\sigma_{i-1}, \sigma_i) = ku_i$. Let $L(\sigma_i)$ denote the 2-dimensional space of $\sigma_i$ (now the negative space). Each $L(\sigma_i) = ku_i \perp ku_{i+1}$ can be viewed as a line in $\mathbf{P}V$. The image of the point $ku_i$ under the map $\pi$ is the point $P(\alpha\sigma_{i-1}, \alpha\sigma_i) = Rv_i$, say, of $\mathbf{P}M$. Then, from the definition of $\pi$, we have $Rv_i = L(\alpha\sigma_{i-1}) \cap L(\alpha\sigma_i)$. Hence $Rv_i + Rv_{i+1} \subseteq L(\alpha\sigma_i)$. Whenever $|i-j| \geq 2$ the product $\sigma_i\sigma_j$ is a 4-involution and since $\alpha$ preserves 4-involutions, it follows that $L(\alpha\sigma_i) \cap L(\alpha\sigma_j) = 0$. Hence $Rv_i \cap Rv_{i+1} = 0$ and, consequently, $L(\alpha\sigma_i) = Rv_i + Rv_{i+1}$. Since each $L(\alpha\sigma_i)$ is a direct summand of $M$, we now have $M = Rv_1 + \cdots + Rv_n$ and $\pi$ carries the projective frame $ku_1, \ldots, ku_n$ of $\mathbf{P}V$ to the projective frame $Rv_1, \ldots, Rv_n$ of $\mathbf{P}M$.

If $P_1$ and $P_2$ are two orthogonal points in $\mathbf{P}V$ whose images $\pi P_1$ and $\pi P_2$ generate a line $\pi P_1 + \pi P_2$ then we may take $P_1 = ku_1$ and $P_2 = ku_2$ as above. Let $P$ be any point on the line $P_1 + P_2$ and choose a minimal pair $(\sigma_1, \tau)$ of 2-involutions, with $\sigma_1$ as before, and $P = P(\sigma_1, \tau)$. Then $\pi P$ lies on $L(\alpha\sigma_1) = \pi P_1 + \pi P_2$ as required. The case $f(P_1, P_2) \neq 0$ can be handled as in Proposition 4.

PROPOSITION 6. *The collineations of Propositions* 4 *and* 5 *are thick.*

PROOF. Let $P_1 = ku_1$ and $P_2 = ku_2$ be points in $\mathbf{P}V$ with images $\pi P_1 = Rv_1$ and $\pi P_2 = Rv_2$ spanning a line in $\mathbf{P}M$. We may assume $f(u_1, u_1) = f(u_2, u_2)$. Consider first $f(u_1, u_2) = 0$ and expand $u_1, u_2$ to an orthogonal basis $u_1, \ldots, u_n$ of $V$. By symmetry $R(v_1 + pv_2)$ is the image of $k(u_1 + u_2)$. If $p$ is a unit then $\pi$ is thick. Assume, therefore, $p \in \mathfrak{m}$. Let $k(u_1 - u_2)$ have image $R(a_1v_1 + a_2v_2)$. Since $u_1 + u_2, u_1 - u_2, u_3, \ldots, u_n$ is an orthogonal basis of $V$ it follows, as before, that $R(v_1 + pv_2)$ and $R(a_1v_1 + a_2v_2)$ must span $Rv_1 + Rv_2$. Hence $a_2$ is a unit which can be made 1. Again we are finished unless $a_1 = q \in \mathfrak{m}$. If $n$ is odd, apply $\alpha$ to the identity $\Psi(u_2)\Psi(u_1 + u_2) = \Psi(u_1 - u_2)\Psi(u_2)$. Then, as in Proposition 4,

$E(U_2, v_2)E(X, v_1 + pv_2) = E(Y, qv_1 + v_2)E(U_2, v_2)$ where $U_2 = \sum_{i \neq 2} Rv_i$. Hence

$$\begin{aligned} E(Y, qv_1 + v_2)(-v_1 + pv_2) &= E(Y, qv_1 + v_2)E(U_2, v_2)(v_1 + pv_2) \\ &= E(U_2, v_2)E(X, v_1 + pv_2)(v_1 + pv_2) \\ &= E(U_2, v_2)(v_1 + pv_2) = -v_1 + pv_2 \end{aligned}$$

and, therefore, $-v_1 + pv_2 \in R(qv_1 + v_2)$. This is impossible since $q \in \mathfrak{m}$. When $n$ is even a similar calculation can be made starting with the 2-involution identity $\sigma_2\Psi(u_3)\Psi(u_1 + u_2) = \Psi(u_1 - u_2)\Psi(u_3)\sigma_2$, where $\sigma_2 = \Psi(u_2)\Psi(u_3)$. The remaining case with $f(u_1, u_2) \neq 0$ is handled as in Proposition 4.

**5. Representations over local rings.** Let $G$ be a subgroup of $U(f)$ containing $SU(f)$ and $\alpha\colon G \to GL(M)$ a representation, with $\alpha SU(f)$ nontrivial, where $M$ is a free module over a commutative local ring $R$ with 2 a unit and $n = \dim f = \operatorname{rank} M \geq 3$. Since $SU(f)$ is perfect, $\alpha SU(f) \subseteq SL(n, R)$. Then except for $n = 4$ or 6 the homomorphism $\alpha$ induces a thick collineation $\pi\colon \mathbf{P}V \to \mathbf{P}M$. The cases $n = 4$ and 6 can be included if the hypotheses are modified to $I(f) \subseteq G$ and $\operatorname{card} \alpha I(f) > 2$. By the Fundamental Theorem there now exists a generalized place $\varphi\colon k \to R \cup \infty$ with valuation ring $A = \varphi^{-1}(R)$ and homomorphism $\varphi\colon A \to R$, a free $A$-module $N$ in $V$ of rank $n$ and a $\varphi$-semilinear map $\beta\colon N \to M$ inducing $\pi$. Thus $\pi P = R\beta(P \cap N)$ for any point $P$ in $\mathbf{P}V$. We study these objects further and use them to describe the original representation $\alpha$ via a twisted congruence subgroup in $G$.

We first establish that $A^* = A$ and $\mathfrak{m}(A)^* = \mathfrak{m}(A)$. The $A$-module $N$ in $V$ is called *maximal* if $N = \{x \in V \mid f(x, x) \in \mathfrak{c}\}$ for some fractional ideal $\mathfrak{c}$ of $A$. Denote by $U(N)$ the subgroup of $U(f)$ consisting of all $\sigma \in U(f)$ such that $\sigma(N) = N$. If $N$ is maximal then $U(N) = U(f)$.

PROPOSITION 7. (i) $A^* = A$ *and* $\mathfrak{m}(A)^* = \mathfrak{m}(A)$.
(ii) *The $A$-module $N$ is maximal and has an orthogonal basis.*
(iii) $U(N) = U(f)$.

PROOF. Let $x, y \in N$ be primitive (that is, $x, y$ are not in $\mathfrak{m}(A)N$) with $\pi(kx) \neq \pi(ky)$. Assume, if possible, $c = 2f(x, x)^{-1}f(x, y)$ is not in $A$, or equivalently, that $f(x, y)^{-1}f(x, x) \in \mathfrak{m}(A)$. Let $\beta(x) = v$ and $\beta(y) = w$ so that $Rv \neq Rw$. Then

$$\Psi(x)\Psi(y) = \Psi(\Psi(x)(y))\Psi(x) = \Psi(y - cx)\Psi(x)$$

is an identity in $SU(f)$. If $n$ is odd, applying the homomorphism $\alpha$ gives an identity $E(X, v)E(Y, w) = E(Z, \varphi(c^{-1})w - v)E(X, v)$ in $SL(M)$. Hence $E(X, v)(w) \in R(\varphi(c^{-1})w - v)$. But $E(X, v)(w) = -w + 2av$ for some $a \in R$ which forces the contradiction $Rv = Rw$. If $n \geq 8$ is even, choose $z$ primitive in $N$ and orthogonal to $x$ and $y$. Then a similar contradiction can be achieved using the 2-involutions $\Psi(x)\Psi(z)$, $\Psi(y)\Psi(z)$ and $\Psi(y - cx)\Psi(z)$. Hence, in both cases, $f(x, y) \in f(x, x)A$.

Since $A$ is a valuation ring the module $N$ has an orthogonal decomposition into indecomposable components of rank one or two. If $N$ has an indecomposable binary component $B = Ax + Ay$, then necessarily $f(x, x) \in f(x, y)\mathfrak{m}(A)$. This gives a contradiction with the previous paragraph since $\beta(x)$, $\beta(y)$ expands to a basis of $M$. Thus $N$ has an orthogonal basis and we may assume $N = Au_1 \perp \cdots \perp Au_n$ and $M = Rv_1 + \cdots + Rv_n$ where $\beta(u_i) = v_i$, $1 \leq i \leq n$. Let $f(u_i, u_i) = a_i$,

$1 \le i \le n$, where each $a_i$ is a nonzero element of $k$. Since $A$ is a valuation ring there exists $a_1$, say, such that $a_1^{-1}a_i \in A$, $1 \le i \le n$.

Let $e \in A$ be a unit. If $e^* \notin A$ replace $e$ by $e^{-1}$. Hence $e^* \in A$. Assume $e^*$ is not a unit, so $e^* \in \mathfrak{m}(A)$. Put $x = e^*u_1 + eu_2$ and $y = u_1$. Then $\pi(kx) \neq \pi(ky)$ and consequently $f(x,y) \in f(x,x)A$. Therefore, $ea_1 \in (ea_1e^* + e^*a_2e)A$ which gives the contradiction $1 \in \mathfrak{m}(A)$. Thus $e^*$ must be a unit. Hence $(1+\mathfrak{m}(A))^* \subseteq A$, so that $\mathfrak{m}(A)^* \subseteq A$. It follows that $A^* = A$ and $\mathfrak{m}(A)^* = \mathfrak{m}(A)$.

Next let $p \in \mathfrak{m}(A)$ and put $x = p^*u_1 + u_i$, $i \neq 1$, and $y = u_1$. Then $\pi(kx) \neq \pi(ky)$ so that $f(x,y) \in f(x,x)A$. Therefore, $pa_1$ lies in the fractional ideal $(pa_1p^* + a_i)A$ and, since $A$ is a valuation ring, $pa_1 \in a_iA$. Thus $a_1\mathfrak{m}(A) \subseteq a_iA$. If $\mathfrak{m}(A)$ is not a principal ideal then $a_1^{-1}a_i$ must be a unit, since $a_1^{-1}a_i \in A$. In this case $N$ is now a modular lattice. If, on the other hand, $\mathfrak{m}(A)$ is a principal ideal then either $a_1^{-1}a_i$ is a unit, or $a_iA = a_1\mathfrak{m}(A)$. The second possibility cannot occur since $f$ represents only one norm class. Thus again $N$ is modular.

Finally, let $x = \sum c_iu_i \in N$ with at least one coefficient, say $c_1$, a unit in $A$. Assume $f(x,x) \in a_1\mathfrak{m}(A)$, so that necessarily a second coefficient, say $c_2$, is also a unit. Put $y = u_2$. Then $\pi(kx) \neq \pi(ky)$ and $f(x,y) \in f(x,x)A$. This forces the contradiction $1 \in \mathfrak{m}(A)$. Hence $f(x,x)A = a_1A$ and $N$ is now necessarily a maximal lattice.

DEFINITION. Let $\chi\colon G \to \mathfrak{u}(R)$ be a character, that is, a homomorphism from $G$ into the group of units $\mathfrak{u}(R)$ of $R$, and $\mathfrak{a}$ the kernel of $\varphi\colon A \to R$. Then $G(\mathfrak{a},\chi)$ denotes the twisted congruence subgroup of the group $G$ consisting of all $\sigma \in G$ such that

(i) $\chi(\sigma) \in \varphi(A)$, so $\chi(\sigma) = \varphi(a_\sigma)$ for some $a_\sigma \in A$,

(ii) $a_\sigma\sigma(x) \equiv x \bmod \mathfrak{a}N$ for all $x \in N$.

Note that the definition is independent of the choice of $a_\sigma$ since $\mathfrak{a}$ is the kernel of $\varphi$ and condition (ii) is modulo $\mathfrak{a}N$.

PROOF OF THEOREM 1. Many parts of the theorem have already been established. It remains to construct a character $\chi\colon G \to \mathfrak{u}(R)$, show that $U(\mathfrak{a},\chi)$ is the kernel of $\alpha$, and describe the action of $\alpha\sigma$ for $\sigma \in G$.

Consider first $n \ge 3$ odd. Let $x \in N$ be primitive and $N = Ax \perp X$. Then the image of $-\Psi(x) \in SU(f)$ under $\alpha$ is $E(Y,\beta(x))$ where $Y = \beta(X)$. For if $w \in X$ is primitive, then $\Psi(x)$ and $\Psi(w)$ commute and hence $E(Y,\beta(x))$ and $E(Z,\beta(w))$ commute. Since $R\beta(x) \cap R\beta(w) = 0$, for otherwise $\alpha(\Psi(x)\Psi(w)) = 1_M$, it follows that $\beta(w) \in Y$. Hence $\beta(X) \subseteq Y$ and, since they have the same rank, $\beta(X) = Y$. For any $\sigma \in G$ and primitive $x \in N$ we have $\sigma(-\Psi(x))\sigma^{-1} = -\Psi(\sigma(x))$ in $G$. Applying the homomorphism $\alpha$ gives $(\alpha\sigma)E(\beta(X),\beta(x))(\alpha\sigma)^{-1} = E(\beta\sigma(X),\beta\sigma(x))$. Hence, from comparing the positive spaces,

$$(\alpha\sigma)(\beta(x)) = \chi(\sigma,x)\beta\sigma(x)$$

for some unit $\chi(\sigma,x)$ in $R$. We show $\chi(\sigma) = \chi(\sigma,x)$ is independent of the choice of $x$. Let $y \in N$ be primitive with $f(x,y) = 0$. Then $x+y \in N$ is also primitive. Moreover, if $\alpha(-\Psi(x)) = E(Y,\beta(x))$, then $\beta(y) \in Y$ and hence $R\beta(x) \cap R\beta(y) = 0$. Then, from $\chi(\sigma,x+y)\beta\sigma(x+y) = (\alpha\sigma)\beta(x+y) = \chi(\sigma,x)\beta\sigma(x) + \chi(\sigma,y)\beta\sigma(y)$ it follows that $\chi(\sigma,x+y) = \chi(\sigma,x) = \chi(\sigma,y)$. Thus $\chi(\sigma,x)$ is independent of $x$ and we have constructed a map $\chi\colon G \to \mathfrak{u}(R)$. It is easily seen that $\chi$ is a group homomorphism. Also, we have established that

$$(\alpha\sigma)(\beta(x)) = \chi(\sigma)\beta\sigma(x).$$

Now let $n \geq 8$ be even. Let $\tau_1 = \Psi(x)\Psi(y)$ and $\tau_2 = \Psi(x)\Psi(z)$ where $x, y, z$ are primitive orthogonal elements of $N$. Then $(\tau_1, \tau_2)$ is a minimal pair of 2-involutions with associated point $P(\tau_1, \tau_2) = L(\tau_1) \cap L(\tau_2) = kx$. From the construction of $\pi$ in Proposition 5 and the definition of $\beta$ we have $L(\alpha\tau_1) = \beta L(\tau_1)$, $L(\alpha\tau_2) = \beta L(\tau_2)$ and $P(\alpha\tau_1, \alpha\tau_2) = \beta P(\tau_1, \tau_2)$. For any $\sigma \in G$, $(\sigma\tau_1\sigma^{-1}, \sigma\tau_2\sigma^{-1})$ remains a minimal pair as also does its image under $\alpha$. Since $L(\sigma\tau_1\sigma^{-1}) = \sigma L(\tau_1)$, $L(\sigma\tau_2\sigma^{-1}) = \sigma L(\tau_2)$ and $P(\sigma\tau_1\sigma^{-1}, \sigma\tau_2\sigma^{-1}) = \sigma P(\tau_1, \tau_2)$ it follows that $(\alpha\sigma)\beta(x) = \chi(\sigma, x)\beta\sigma(x)$ for some unit $\chi(\sigma, x)$ in $R$. Again it follows that $\chi(\sigma, x)$ is independent of $x$ and that $\chi$ induces a group homomorphism from $G$ to the units of $R$.

If $\mathfrak{a}$ is the kernel of the homomorphism $\varphi\colon A \to R$ then clearly $\mathfrak{a}N$ is the kernel of the $\varphi$-semilinear map $\beta\colon N \to M$. Finally we must show that the twisted congruence subgroup $G(\mathfrak{a}, \chi)$ is the kernel of $\alpha$. Let $\sigma \in \ker\alpha$. Then $\alpha(\sigma) = 1_M$ and consequently $\beta(x) = \chi(\sigma)\beta\sigma(x)$ for any $x \in N$. Hence $\chi(\sigma) \in \varphi(A)$. Therefore, $\beta(x - a_\sigma\sigma(x)) = 0$ where $\varphi(a_\sigma) = \chi(\sigma)$. Thus $\sigma \in G(\mathfrak{a}, \chi)$. The converse is similar. This completes the proof of Theorem 1.

REMARK. It is quite possible for $\mathfrak{a} = 0$ and $\varphi$ and $\beta$ to be injections. The kernel of $\alpha$ is then contained in the center of $U(f)$.

PROPOSITION 8. *Let $\varphi\colon k \to R \cup \infty$ be the generalized place in Theorem* 1 *and $A = \varphi^{-1}(R)$ the associated valuation ring. Let $K$ be the fixed field of $k$ under the involution* $^*$. *Then*

(i) $\mathbf{Q} \subseteq A$,

(ii) *if* $\mathbf{R} \subseteq K$ *then* $\mathbf{R} \subseteq A$,

(iii) *$B/\mathfrak{m}(B)$ is formally real, where $B = K \cap A$ is the induced valuation ring of $K$.*

PROOF. Since $\varphi(1) = 1$ we know $\mathbf{Z} \subseteq A$. Assume $\varphi(p) \in \mathfrak{m}(R)$ for some odd prime $p$ in $\mathbf{Z}$. Then $p \in \mathfrak{m}(A)$. Let $N = Au_1 \perp \cdots \perp Au_n$ be the maximal lattice in Theorem 1 where we may assume $f(u_i, u_i) = a \neq 0$, $1 \leq i \leq n$. Since the quadratic form $\langle 1, 1, \ldots, 1\rangle$ is isotropic over the finite field $\mathbf{F}_p$ there exist integers $c_1, \ldots, c_n$ with $c_1 = 1$ such that $\sum c_i^2 \in p\mathbf{Z}$. Then $x = \sum c_i u_i \in N$ is primitive and $f(x, x) \in a\mathfrak{m}(A)$, since the involution is trivial on $\mathbf{Z}$. This contradicts the fact that $N$ is a maximal lattice (Proposition 7). Hence $\varphi(p)$ is a unit in $R$ for all odd primes $p$. By hypothesis 2 is a unit in $R$ and hence also in $A$. It follows that $\mathbf{Q} \subseteq A$ and the valuation induced by $\varphi$ on $\mathbf{Q}$ is trivial.

Now assume $\mathbf{R} \subseteq K$. Let $a \in \mathbf{R}$ with $0 < a < 1$. Assume $\varphi(a) = \infty$ so that $a^{-1} \in \mathfrak{m}(A)$. Let $a^{-1} - 1 = b^2$ where $b \in \mathbf{R}$. Then $x = bu_1 + u_2$ is primitive in $N$ but $f(x, x) \in a\mathfrak{m}(A)$, contradicting the fact that $N$ is maximal. Therefore, $\mathbf{R} \subseteq A$ by the Archimedean axiom on $\mathbf{R}$, and the induced valuation on $\mathbf{R}$ is now trivial. The proof of (iii) is similar.

REMARK. Proposition 8(ii) is more general. The same result holds for any subfield of $\mathbf{R}$ all of whose positive elements are squares. If $k$ is such a field then Theorem 1 suggests that the only nontrivial normal subgroup of the orthogonal group $SO(f)$, $n \neq 4$, is its center. This is true, see Bröcker [**4**]. A similar result should hold for unitary groups $SU(f)$ over $k(i)$, $i^2 = -1$.

**6. Representations over commutative rings.** We now extend the results of the previous section to representations $\alpha\colon G \to GL(n, R)$, where $R$ is a commutative ring with 2 not a zero divisor. If 2 is not a unit in $R$, enlarge $R$ by localizing with respect to the multiplicative set generated by 2. Assume, therefore, 2 is a unit in $R$. Let $\mathfrak{p}$ be a prime ideal of $R$, $R_\mathfrak{p}$ the localization at $\mathfrak{p}$ and $\varepsilon_\mathfrak{p}\colon R \to R_\mathfrak{p}$ the canonical homomorphism. Localize $M$ at $\mathfrak{p}$ and let $M \to M_\mathfrak{p}$ be the natural extension of $\varepsilon_\mathfrak{p}$. Then we have a homomorphism $\eta_\mathfrak{p}\colon GL(M) \to GL(M_\mathfrak{p})$ from the group $GL(M) = GL(n, R)$ to the group $GL(M_\mathfrak{p}) = GL(n, R_\mathfrak{p})$.

PROOF OF THEOREM 2. Let $\alpha\colon SU(f) \to GL(M)$ be a nontrivial representation of $SU(f)$. Then Theorem 1 can be applied to the composite map $\eta_\mathfrak{p} \circ \alpha\colon SU(f) \to GL(n, R_\mathfrak{p})$. If the image of $SU(f)$ is now trivial then obviously the kernel is $SU(f)$. Otherwise the kernel is a twisted congruence subgroup $SU(\mathfrak{a}_\mathfrak{p}, \chi_\mathfrak{p})$ of $SU(f)$. Let $H$ be the intersection of all these twisted congruence subgroups for which $\eta_\mathfrak{p} \circ \alpha$ is nontrivial as $\mathfrak{p}$ varies over the maximal ideals of $R$. Clearly $H$ is a normal subgroup of $SU(f)$ containing the kernel of $\alpha$. Conversely, let $\sigma \in H$. Then $\eta_\mathfrak{p}(\alpha\sigma) = 1$ for all maximal ideals $\mathfrak{p}$. Hence $(\alpha\sigma(x) - x)_\mathfrak{p} = 0$ in $M_\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ and all $x \in M$. It follows (for example, Bass [**2**, p. 108]) that $(\alpha\sigma)(x) = x$ in $M$ for all $x \in M$. Thus $\sigma \in \ker\alpha$ and

$$1 \to H \to SU(f) \xrightarrow{\alpha} GL(n, R)$$

is an exact sequence. Finally, since $SU(f)$ is perfect, the characters $\chi_\mathfrak{p}\colon SU(f) \to \mathfrak{u}(R_\mathfrak{p})$ are all trivial.

To prove Theorem 3 we need to study the generalized places $\varphi\colon \mathbf{C}(X) \to R \cup \infty$, with $R$ a local ring, extending an injection $\varphi\colon \mathbf{C} \to R$. Recall the notation $A_c$ and $\varphi_{c,m}$ from the introduction.

PROPOSITION 9. *Let $\varphi\colon \mathbf{C}(X) \to R \cup \infty$ be a generalized place, which induces the trivial valuation on* $\mathbf{C}$, *and $A = \varphi^{-1}(R)$ the associated valuation ring. Then one of the following occurs*:

(i) $A = \mathbf{C}(X)$ *and $\varphi$ is an injection into* $R$,

(ii) $A = A_c$ *and $\varphi = \varphi_{c,m}$ for some $c \in \mathbf{C}$ and integer $m \geq 1$*,

(iii) $A = A_\infty$ *and $\varphi = \varphi_{\infty,m}$ for some integer $m \geq 1$*.

PROOF. We have $\varphi(\mathbf{C}) = \mathbf{C} \subseteq R$. Let $\mathfrak{a}$ be the kernel of $\varphi\colon A \to R$. Assume first $\varphi(X) = x \in R$. Then $\mathbf{C}[x] \subseteq R$. If $g(x)$ is a unit in $R$ for all nonzero polynomials $g(X) \in \mathbf{C}[X]$ then $A = \mathbf{C}(X)$, $\mathfrak{a} = 0$ and $\varphi$ is an injection. Assume, therefore, there exists a monic nonconstant polynomial $g(X) \in \mathbf{C}[X]$ with $g(x) \in \mathfrak{m}(R)$. Let $g(X)$ have minimal degree and $c \in \mathbf{C}$ be a root of $g(X)$. Then $g(X) = X - c$. If $h(X) \in \mathbf{C}[X]$ is a polynomial relatively prime to $g(X)$, then $h(x)$ must be a unit of $R$, for otherwise $1 \in \mathfrak{m}(R)$. Hence $A = A_c$ and $\mathfrak{m}(A) = (X - c)A_c$. Since the valuation ring $A_c$ is discrete, the kernel $\mathfrak{a} = (X - c)^m A_c$ for some integer $m \geq 1$. Then $\varphi = \varphi_{c,m}$. Finally assume $\varphi(X) = \infty$. Then $X^{-1} \in \mathfrak{m}(A)$ and $\varphi(X^{-1}) = x \in \mathfrak{m}(R)$. Hence $\varphi(X^{-1} - c)$ is a unit of $R$ for all nonzero $c \in \mathbf{C}$. Therefore, $(X - a)/(X - b)$ is a unit in $A$ for all $a, b \in \mathbf{C}$. Consequently $A = A_\infty$ consists of all rational functions with degree $\leq 0$. Hence $\mathfrak{m}(A) = X^{-1}A_\infty$ and the kernel $\mathfrak{a} = X^{-m}A_\infty$ for some $m \geq 1$. Thus $\varphi = \varphi_{\infty,m}$.

PROOF OF THEOREM 3. We apply Theorems 1 and 2 to the special case $k = \mathbf{C}(X)$. Let $\varphi\colon \mathbf{C}(X) \to R \cup \infty$ be a generalized place occurring in Theorem

1(i). By Proposition 8, $\varphi(\mathbf{R}) = \mathbf{R} \subseteq R$ (after identifying $\mathbf{R}$ with a subfield of $R$). Also $\varphi(i)^2 = \varphi(-1) = -1$ and hence $\varphi(\mathbf{C}) = \mathbf{C} \subseteq R$. Therefore $\varphi$ is one of the generalized places in Proposition 9. However, if $\varphi = \varphi_{c,m}$ with $c \in \mathbf{C}$, then $c \in \mathbf{R}$ since by Theorem 1(i) we must have $A_c^* = A_c$. Let $H$ be a nontrivial normal subgroup of $SU(f)$ as in Theorem 3. By Theorem 2, $H = \bigcap SU(\mathfrak{a}_j, 1)$ where the $\mathfrak{a}_j$ are kernels of generalized places $\varphi_j\colon \mathbf{C}(x) \to R_j \cup \infty$. We may assume $\mathfrak{a}_j \neq 0$ and hence any injective $\varphi_j$ can be ignored. Only a finite number of $\mathfrak{a}_j$ can be nonzero, for otherwise $H = 1$ is trivial. Let $A$ be the intersection of the finite number of valuation rings $A_c$, $c \in \mathbf{R} \cup \infty$, associated with generalized places with nontrivial kernel, and $\mathfrak{a}$ the intersection of these kernels. Then $\mathfrak{a}$ is an ideal of $A$. By the weak approximation theorem for valuations we can choose a common basis $u_1, \ldots, u_n$ for all the associated maximal lattices in Theorem 1(ii). Then $N = Au_1 + \cdots + Au_n$ is the intersection of these maximal lattices and hence $\sigma(N) = N$ for all $\sigma \in SU(f)$. Let $SU(\mathfrak{a})$ be the congruence subgroup of all $\sigma \in SU(f)$ with $(\sigma - 1)N \subseteq \mathfrak{a}N$. Then $H = \bigcap SU(\mathfrak{a}_j, 1) = SU(\mathfrak{a})$, completing the proof.

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